A REFINEMENT OF THE COMPLEX CONVEXITY THEOREM VIA SYMPLECTIC TECHNIQUES

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Abstract. We apply techniques from symplectic geometry to extend and give a new proof of the complex convexity theorem of Gindikin-Krötz.

1. Introduction

Let G be a semisimple Lie group and $G = NAK$ be an Iwasawa decomposition. Let $X \in \mathfrak{a} = \mathrm{Lie}(A)$ be such that $\mathrm{Spec}(\mathrm{ad}X) \subseteq]-\frac{\pi}{2}, \frac{\pi}{2}[$ and set $a = \exp(iX)$. It is known that $Ka \subseteq N_{\mathbb{C}}A_{\mathbb{C}}K_{\mathbb{C}}$. The main result of this paper asserts that

$$\Im \log \tilde{a}(Ka) = \mathrm{conv}(\mathcal{W}.X) \ . \tag{1.1}$$

Here $\tilde{a} : N_{\mathbb{C}}A_{\mathbb{C}}K_{\mathbb{C}} \to A_{\mathbb{C}}$ denotes a middle projection and $\mathrm{conv}(\mathcal{W}.X)$ stands for the convex hull of the Weyl group orbit $\mathcal{W}.X$. We note that the inclusion $'' \subseteq''$ in (1.1) is the complex convexity theorem from [4].

We prove (1.1) with symplectic methods. As a byproduct we obtain interesting new classes of compact Hamiltonian manifolds and Lagrangian submanifolds thereof.

Let us now be more specific about the used techniques. We first consider the case where G is a complex group. In this situation we show that $B_{\mathbb{C}} = A_{\mathbb{C}}N_{\mathbb{C}}$ carries a natural structure of a Poisson Lie group. Locally, we can identify $B_{\mathbb{C}}$ inside of $G_{\mathbb{C}}/K_{\mathbb{C}}$ and consequently we obtain a local action of $G_{\mathbb{C}}$ on $B_{\mathbb{C}}$. Within this identification the symplectic leaf P_a through $a = \exp(iX) \in B_{\mathbb{C}}$ becomes a local $K_{\mathbb{C}}$-orbit. Interestingly, the symplectic form on P_a remains non-degenerate on the totally real K-orbit $M_a \subseteq P_a$. We then exhibit a Hamiltonian torus action on the compact symplectic manifold M_a and show that that (1.1) becomes a consequence of the Atiyah-Guillemin-Sternberg convexity theorem. Finally, the general case of arbitrary G can be handled by descent to certain Lagrangian submanifolds $Q_a \subseteq M_a$ by means of the recently discovered convexity theorem [6].

2. Notation and basic facts

In this section we recall some basic facts about semisimple Lie algebras and groups. We make an emphasis on the complexified Iwasawa decomposition. Furthermore we review some standard facts on the double complexification of a semisimple Lie algebra.

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2.1. **Iwasawa decomposition and complex crown.** We let $\mathfrak{g}$ denote a semisimple Lie algebra and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$. For a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a}) \subseteq \mathfrak{a}^*$ be the corresponding root system. Then $\mathfrak{g}$ admits a root space decomposition

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}^\alpha,$$

where $\mathfrak{m} = \mathfrak{z}_\mathfrak{k}(\mathfrak{a})$ and $\mathfrak{g}^\alpha = \{X \in \mathfrak{g} : (\forall H \in \mathfrak{a})\ [H, X] = \alpha(H)X\}$.
For a fixed positive system Σ^+ define $\mathfrak{n} = \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}^\alpha$. Then we have the Iwasawa decomposition on the Lie algebra level:

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}.$$

For any real Lie algebra $\mathfrak{l}$ we write $\mathfrak{l}_\mathbb{C}$ for its complexification.

In the following $G_\mathbb{C}$ will denote a simply connected Lie group with Lie algebra $\mathfrak{g}_\mathbb{C}$. We write $G, K, K_\mathbb{C}, A, A_\mathbb{C}, N$ and $N_\mathbb{C}$ for the analytic subgroups of $G_\mathbb{C}$ corresponding to the subalgebras $\mathfrak{g}, \mathfrak{k}, \mathfrak{k}_\mathbb{C}, \mathfrak{a}, \mathfrak{a}_\mathbb{C}, \mathfrak{n}$ and $\mathfrak{n}_\mathbb{C}$, respectively.
The Weyl group of Σ can be defined by $\mathcal{W} = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$.

Following [1] we define a bounded and convex subset of $\mathfrak{a}$ that plays a central role:

$$\Omega = \{X \in \mathfrak{a} : |\alpha(X)| < \frac{\pi}{2}\ \forall\alpha \in \Sigma\}. \tag{2.1}$$

With Ω we define a $G - K_\mathbb{C}$-double coset domain in $G_\mathbb{C}$ by

$$\tilde{\Xi} = G \exp(i\Omega)K_\mathbb{C}\ .$$

Also we write

$$\Xi = \tilde{\Xi}/K_\mathbb{C}$$

for the union of right $K_\mathbb{C}$-cosets of $\tilde{\Xi}$ in the complex symmetric space $G_\mathbb{C}/K_\mathbb{C}$. We refer to Ξ as the *complex crown* of the symmetric space G/K. Notice that Ξ is independent of the choice of $\mathfrak{a}$, hence generically defined through G/K.
It is known that $\tilde{\Xi}$ is an open and G-invariant subset of $N_\mathbb{C}A_\mathbb{C}K_\mathbb{C}$ (cf. [7] for a short proof).

Next we consider the open and dense cell $N_\mathbb{C}A_\mathbb{C}K_\mathbb{C} \subseteq G_\mathbb{C}$ in more detail. Define $F = K_\mathbb{C} \cap A_\mathbb{C}$ and recall that $F = K \cap \exp(i\mathfrak{a})$ is a finite 2-group. Standard techniques imply that the mapping

$$N_\mathbb{C} \times [A_\mathbb{C} \times_F K_\mathbb{C}] \to N_\mathbb{C}A_\mathbb{C}K_\mathbb{C}, \quad (n, [a, k]) \mapsto nak \tag{2.2}$$

is a biholomorphism. In particular (2.2) induces a holomorphic map $\tilde{n} : N_\mathbb{C}A_\mathbb{C}K_\mathbb{C} \to N_\mathbb{C}$ and a multi-valued holomorphic mapping $\tilde{a} : N_\mathbb{C}A_\mathbb{C}K_\mathbb{C} \to A_\mathbb{C}$ such that $x \in \tilde{n}(x)\tilde{a}(x)K_\mathbb{C}$ for all $x \in N_\mathbb{C}A_\mathbb{C}K_\mathbb{C}$.

Set $B_\mathbb{C} = N_\mathbb{C}A_\mathbb{C}$. Clearly, $\mathfrak{b}_\mathbb{C} = \mathfrak{a}_\mathbb{C} + \mathfrak{n}_\mathbb{C}$ is the Lie algebra of $B_\mathbb{C}$. We define a multi-valued holomorphic map $\tilde{b} : N_\mathbb{C}A_\mathbb{C}K_\mathbb{C} \to B_\mathbb{C}$ by $\tilde{b}(x) = \tilde{n}(x)\tilde{a}(x)$.

Recall that Ξ is contractible, and hence simply connected. It follows that the restriction $\tilde{a}|_{\tilde{\Xi}}$ has a unique single-valued holomorphic lift $\log \tilde{a} : \tilde{\Xi} \to \mathfrak{a}_\mathbb{C}$ such that $\log \tilde{a}(\mathbf{1}) = 0$. Consequently, $\tilde{b}|_{\tilde{\Xi}}$ lifts to a single-valued holomorphic map $\tilde{\Xi} \to B_\mathbb{C}$ which shall also be denoted by $\tilde{b}$.

2.2. **Double complexification of a Lie algebra.** For the remainder of this section we will assume that $\mathfrak{g}$ carries a complex structure, say j. Then $\mathfrak{g}$ can be viewed as the complexification of its compact real form $\mathfrak{k}$ and the Cartan decomposition becomes $\mathfrak{g} = \mathfrak{k} + j\mathfrak{k}$. The Cartan involution θ on $\mathfrak{g}$ coincides with the complex conjugation $^{-}: \mathfrak{g} \to \mathfrak{g}$ with respect to $\mathfrak{k}$.
A second complexification yields $\mathfrak{g}_{\mathbb{C}}$, which carries another complex structure $i : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$.
The following map φ defines a real Lie algebra isomorphism.

$$\varphi : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g} \times \mathfrak{g}, \quad X + iY \mapsto (X + jY, \bar{X} + j\bar{Y}).$$

Under φ we have the following identifications:

$$\mathfrak{k}_{\mathbb{C}} = \{(Z, Z) : Z \in \mathfrak{g}\}, \tag{2.3}$$

$$\mathfrak{a}_{\mathbb{C}} = \{(Z, -Z) : Z \in \mathfrak{a} + j\mathfrak{a}\}, \tag{2.4}$$

$$\mathfrak{n}_{\mathbb{C}} = \{(X + jY, \theta(X - jY)) : X, Y \in \mathfrak{n}\} = \mathfrak{n} \times \bar{\mathfrak{n}}, \tag{2.5}$$

where $\bar{\mathfrak{n}} = \bigoplus_{-\alpha \in \Sigma^+} \mathfrak{g}^{\alpha}$.

3. The complex convexity theorem for complex groups

The objective of this section is to provide a symplectic proof of the complex convexity theorem in the case of G complex. If G is complex, then we can endow $B_{\mathbb{C}}$ with a natural structure of a Poisson Lie group. The symplectic leaves become local $K_{\mathbb{C}}$-orbits. We show that the totally real K-orbit in each leaf is again a symplectic manifold. With that we obtain compact symplectic manifolds with appropriate Hamiltonian torus actions. The complex convexity theorem then becomes a consequence of the Atiyah-Guillemin-Sternberg convexity theorem.

3.1. **The Poisson Lie group $B_{\mathbb{C}}$.** Throughout this section $\mathfrak{g}$ denotes a complex semisimple Lie algebra. Our first task is to define a bilinear form $\langle , \rangle$ on $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \times \mathfrak{g}$ which gives $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}})$ the structure of a Manin triple. To that end let κ be the Killing form of the complex Lie algebra $\mathfrak{g}$. Then define on $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \times \mathfrak{g}$ a bilinear form by

$$\langle (X, Y), (X', Y') \rangle = \Re\kappa(X, X') - \Re\kappa(Y, Y')$$

for $(X, Y), (X', Y') \in \mathfrak{g}_{\mathbb{C}}$.

Proposition 3.1. *The bilinear form $\langle , \rangle$ on $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \times \mathfrak{g}$ has the following properties:*
(i) $\langle , \rangle$ *is symmetric, non-degenerate and $G_{\mathbb{C}}$-invariant.*
(ii) $\langle \mathfrak{b}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}} \rangle = \{0\}$ *and* $\langle \mathfrak{k}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}} \rangle = \{0\}$.

Proof. (i) is immediate from the definition. Moving on (ii) we observe that the relations $\langle \mathfrak{k}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}} \rangle = \{0\}$ and $\langle \mathfrak{a}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}} \rangle = \{0\}$ are straightforward from the identifications (2.3)-(2.5). Finally, the fact that root spaces $\mathfrak{g}^{\alpha}$ and $\mathfrak{g}^{\beta}$ are κ-orthogonal if $\alpha + \beta \neq 0$ implies $\langle \mathfrak{n}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}} + \mathfrak{n}_{\mathbb{C}} \rangle = \{0\}$. □

Notice that Proposition 3.1 just says that $\langle , \rangle$ turns $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}})$ into a Manin triple. Accordingly $B_{\mathbb{C}}$ becomes a Poisson Lie group whose symplectic leaves are local $K_{\mathbb{C}}$-orbits (cf. [3]). Here we shall only be interested in the leaves through points $a = \exp(iX)$ for $X \in \Omega$. Denote the leaf containing a by P_a. Then

$$P_a = \{\tilde{b}(ka) \in B_{\mathbb{C}} : \ k \in K_{\mathbb{C}}, ka \in N_{\mathbb{C}} A_{\mathbb{C}} K_{\mathbb{C}}\}_0 \, ,$$

where $\{\cdot\}_0$ refers to the connected component of $\{\cdot\}$ containing a.

For an element $Z \in \mathfrak{k}_\mathbb{C}$ we write $\tilde{Z}$ for the corresponding vector field on P_a, i.e. if $b = \tilde{b}(ka) \in P_a$, then

$$\tilde{Z}_b = \frac{d}{dt}\Big|_{t=0} \tilde{b}(\exp(tZ)ka) \ .$$

Write $\mathbf{pr}_{\mathfrak{k}_\mathbb{C}} : \mathfrak{g}_\mathbb{C} \to \mathfrak{k}_\mathbb{C}$ and $\mathbf{pr}_{\mathfrak{b}_\mathbb{C}} : \mathfrak{g}_\mathbb{C} \to \mathfrak{b}_\mathbb{C}$ for the projections along $\mathfrak{b}_\mathbb{C}$, resp. $\mathfrak{k}_\mathbb{C}$.

We notice that $T_bP_a = \{\tilde{Z}_b : Z \in \mathfrak{k}_\mathbb{C}\}$. The symplectic form $\tilde{\omega}$ on P_a is then given by

(3.1)
$$\tilde{\omega}_b(\tilde{Y}_b, \tilde{Z}_b) = \langle \mathbf{pr}_{\mathfrak{k}_\mathbb{C}}(\mathrm{Ad}(b^{-1})Y), \mathrm{Ad}(b^{-1})Z\rangle = -\langle \mathbf{pr}_{\mathfrak{b}_\mathbb{C}}(\mathrm{Ad}(b^{-1})Y), \mathrm{Ad}(b^{-1})Z\rangle \ ,$$

for $Y, Z \in \mathfrak{k}_\mathbb{C}$ (cf. ([3]). The second equality in (3.1) follows from Proposition 3.1.

3.2. **The totally real K-orbit in the symplectic leaf.** Our interest is not so much with P_a as it is with its totally real submanifold

$$M_a = \tilde{b}(Ka) \ .$$

Then $\tilde{\omega}$ induces a closed 2-form ω on M_a by $\omega = \tilde{\omega}|_{TM_a \times TM_a}$. A priori it is not clear that ω is non-degenerate, i.e. that (M_a, ω) is a symplectic manifold. This will be shown now. We start with a simple algebraic fact.

Lemma 3.2. *With respect to $\langle , \rangle$ one has $\mathfrak{k}^\perp = \mathfrak{k}_\mathbb{C} + \mathfrak{b}$.*

Proof. Because of the non-degeneracy of $\langle , \rangle$, it is sufficient to verify that $\mathfrak{k}_\mathbb{C} + \mathfrak{b} \subseteq \mathfrak{k}^\perp$. Clearly, $\mathfrak{k}_\mathbb{C} \subseteq \mathfrak{k}^\perp$ since $\mathfrak{k}_\mathbb{C}$ is isotropic. As $\mathfrak{b} = \mathfrak{a} + \mathfrak{n}$, it thus remains to show that $\langle \mathfrak{k}, \mathfrak{a}\rangle = \{0\}$ and $\langle \mathfrak{k}, \mathfrak{n}\rangle = \{0\}$. Now $\langle \mathfrak{k}, \mathfrak{a}\rangle = \{0\}$ follows from (2.3-4) and $\Re\kappa(\mathfrak{k}, \mathfrak{a}) = \{0\}$. Finally we show that $\langle \mathfrak{k}, \mathfrak{n}\rangle = \{0\}$. For that fix an arbitrary element $W = (\sum_{\alpha\in\Sigma^+} Y_\alpha, \sum_{\alpha\in\Sigma^+} \bar{Y}_\alpha)$ of $\mathfrak{n} \subseteq \mathfrak{g} \times \mathfrak{g}$; here $Y_\alpha \in \mathfrak{g}^\alpha$. Likewise let $U = (V + \sum_{\alpha\in\Sigma^+}(Z_\alpha + \bar{Z}_\alpha), V + \sum_{\alpha\in\Sigma^+}(Z_\alpha + \bar{Z}_\alpha))$ be an element of $\mathfrak{k} \subseteq \mathfrak{g} \times \mathfrak{g}$; here $V \in \mathfrak{m}$ and $Z_\alpha \in \mathfrak{g}^\alpha$.

As $\mathfrak{m} \perp_\kappa \mathfrak{g}^\alpha$ and $\mathfrak{g}^\alpha \perp_\kappa \mathfrak{g}^\beta$ for $\alpha + \beta \neq 0$, we obtain

$$\begin{aligned}
\langle U, W\rangle &= \sum_{\alpha\in\Sigma^+} \Re\kappa(Z_\alpha + \bar{Z}_\alpha, Y_\alpha) - \Re\kappa(Z_\alpha + \bar{Z}_\alpha, \bar{Y}_\alpha) \\
&= \sum_{\alpha\in\Sigma^+} \Re\kappa(\bar{Z}_\alpha, Y_\alpha) - \Re\kappa(Z_\alpha, \bar{Y}_\alpha) = 0 \ .
\end{aligned}$$

This concludes the proof of the lemma. □

Lemma 3.3. *At any point $b \in M_a$, the bilinear form $\omega_b : T_bM_a \times T_bM_a \to \mathbb{R}$ is non-degenerate.*

Proof. Let $b = \tilde{b}(ka) \in M_a$. Then $b = kak'$ for some $k \in K, k' \in K_\mathbb{C}$. Notice that $T_bM_a = \{\tilde{Y}_b : Y \in \mathfrak{k}\}$.

Assume that there is an $U \in \mathfrak{k}$ such that $\omega_b(\tilde{U}_b, \tilde{Y}_b) = 0$ for all $Y \in \mathfrak{k}$, i.e.

$$\langle \mathbf{pr}_{\mathfrak{k}_\mathbb{C}}(\mathrm{Ad}(b^{-1})U), \mathrm{Ad}(b^{-1})Y\rangle = 0 \quad \forall\ Y \in \mathfrak{k}.$$

We have to show that $\tilde{U}_b = 0$. Set $Z = \mathbf{pr}_{\mathfrak{k}_\mathbb{C}}(\mathrm{Ad}(b^{-1})U)$. Then Ad(b)Z $\in \mathfrak{k}^\perp$. Thus Lemma 3.2 implies that

$$Z \in \mathrm{Ad}(b^{-1})(\mathfrak{k}_\mathbb{C} + \mathfrak{b}). \tag{3.2}$$

On the other hand, by definition,

$$Z \in \mathrm{Ad}(b^{-1})\mathfrak{k} + \mathfrak{b}_\mathbb{C} = \mathrm{Ad}(b^{-1})(\mathfrak{k} + \mathfrak{b}_\mathbb{C}). \tag{3.3}$$

From (3.2) and (3.3) it follows that

$$Z \in \mathrm{Ad}(b^{-1})(\mathfrak{k}_{\mathbb{C}} + \mathfrak{b}) \cap \mathrm{Ad}(b^{-1})(\mathfrak{k} + \mathfrak{b}_{\mathbb{C}}) = \mathrm{Ad}(b^{-1})\mathfrak{g}.$$

Moreover, since Z is an image point of $\mathbf{pr}_{\mathfrak{k}_{\mathbb{C}}}$,

$$Z \in \mathrm{Ad}(b^{-1})\mathfrak{g} \cap \mathfrak{k}_{\mathbb{C}},$$

i.e.

$$\mathrm{Ad}(b)Z \in \mathfrak{g} \cap \mathrm{Ad}(b)\mathfrak{k}_{\mathbb{C}}.$$

As $b = kak'$ we now get

$$\mathrm{Ad}(b)Z \in \mathfrak{g} \cap \mathrm{Ad}(k)\mathrm{Ad}(a)\mathfrak{k}_{\mathbb{C}} = \mathrm{Ad}(k)(\mathfrak{g} \cap \mathrm{Ad}(a)\mathfrak{k}_{\mathbb{C}}) \ .$$

Using standard techniques (see [1] or [7], Lemma 2), it follows from (2.1) that

$$\mathfrak{g} \cap \mathrm{Ad}(a)\mathfrak{k}_{\mathbb{C}} = \mathfrak{z}_{\mathfrak{k}}(X) \ .$$

Hence $\mathrm{Ad}(b)Z \in \mathrm{Ad}(k)\mathfrak{z}_{\mathfrak{k}}(X)$.

From

$$\mathrm{Ad}(b^{-1})U \in Z + \mathfrak{b}_{\mathbb{C}}$$

we conclude

$$U \in \mathrm{Ad}(b)Z + \mathfrak{b}_{\mathbb{C}} \subseteq \mathrm{Ad}(k)\mathfrak{z}_{\mathfrak{k}}(X) + \mathfrak{b}_{\mathbb{C}}.$$

Since U was assumed to lie in $\mathfrak{k}$, we finally get

$$U \in \mathrm{Ad}(k)\mathfrak{z}_{\mathfrak{k}}(X).$$

But this just means that $\tilde{U}_b = 0$, concluding the proof that ω_b is non-degenerate. $\square$

3.3. **Hamiltonian torus action on the totally real leaf M_a.** It follows from Lemma 3.3 that (M_a, ω) is a compact symplectic manifold. Clearly, the torus $T = \exp(j\mathfrak{a})$ acts on M_a as K does. We wish to show that the action of T on M_a is Hamiltonian and identify the corresponding moment map $\Phi : M_a \to \mathfrak{t}^*$, where $\mathfrak{t} = j\mathfrak{a}$ is the Lie algebra of T. We will identify $\mathfrak{t}^*$ with $\mathfrak{a}$ via the linear isomorphism

$$\mathfrak{a} \to \mathfrak{t}^*, \quad Y \mapsto (Z \mapsto \langle iY, Z\rangle) \,. \tag{3.4}$$

We shall write $\mathbf{pr}_{\mathfrak{a}_{\mathbb{C}}} : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{a}_{\mathbb{C}}$ for the projection along $\mathfrak{k}_{\mathbb{C}} + \mathfrak{n}_{\mathbb{C}}$.

Proposition 3.4. *The action of the torus $T = \exp(j\mathfrak{a})$ on M_a is Hamiltonian with momentum map*

$$\Phi : M_a \to \mathfrak{t}^* \simeq \mathfrak{a}, \ \tilde{b}(ka) \mapsto \Im \log \tilde{a}(ka).$$

Proof. We first show that $T = \exp(j\mathfrak{a})$ acts on M_a by symplectomorphisms. For that we first notice that T normalizes $B_{\mathbb{C}}$. Hence the action of T on $M_a \subseteq B_{\mathbb{C}}$ is given by conjugation, i.e.

$$T \times M_a \to M_a, \quad (t, b) \mapsto t.b = tbt^{-1} \,.$$

Moreover, for each $t \in T$ the map $\mathrm{Ad}(t)$ commutes both with $\mathbf{pr}_{\mathfrak{k}_{\mathbb{C}}}$ and $\mathbf{pr}_{\mathfrak{b}_{\mathbb{C}}}$. Combining these facts, it is then straightforward from (3.1) that T acts indeed symplectically on M_a.

Next we show that

$$\omega(\tilde{Y}, \tilde{Z}) = 0 \qquad \forall Y, Z \in \mathfrak{t} \,. \tag{3.5}$$

Fix $b \in M_a$. From the definition (3.1) we obtain that

$$\omega_b(\tilde{Y}_b, \tilde{Z}_b) = \langle \mathbf{pr}_{\mathfrak{b}_{\mathbb{C}}}(\mathrm{Ad}(b^{-1})Y), \mathrm{Ad}(b^{-1}Z)\rangle \ .$$

Now for $Y, Z \in \mathfrak{t}$ we have $\mathrm{Ad}(b^{-1})Y \in Y + \mathfrak{n}_{\mathbb{C}}$ and $\mathrm{Ad}(b^{-1})Z \in Z + \mathfrak{n}_{\mathbb{C}}$. From Proposition 3.1 we know $\langle \mathfrak{n}_{\mathbb{C}}, \mathfrak{n}_{\mathbb{C}} \rangle = \{0\}$ and $\langle \mathfrak{k}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}} \rangle = \{0\}$. Hence, to prove (3.5) it suffices to show $\langle \mathfrak{t}, \mathfrak{n}_{\mathbb{C}} \rangle = \{0\}$.
Let $U = (jZ, jZ) \in \mathfrak{t}$ and $V = (\sum_{\alpha\in\Sigma^+} X_\alpha + jY_\alpha, \sum_{\alpha\in\Sigma^+} \bar{X}_\alpha + j\bar{Y}_\alpha) \in \mathfrak{n}_{\mathbb{C}}$, where $Z \in \mathfrak{a}$ and $X_\alpha, Y_\alpha \in \mathfrak{g}^\alpha$. Then,

$$\langle U, V \rangle = \sum_{\alpha\in\Sigma^+} \Re j\kappa(Z, X_\alpha + jY_\alpha) - \Re j\kappa(Z, \bar{X}_\alpha + j\bar{Y}_\alpha) = 0,$$

since $\mathfrak{g}^\alpha \perp_\kappa \mathfrak{g}^\beta$ for $\alpha + \beta \neq 0$.

Having established (3.5) the symplectic action of T on M_a will be Hamiltonian with moment map Φ if $\iota(\tilde{Z})\omega = d\Phi_Z$ holds for all $Z \in \mathfrak{t}$. Fix $b \in M_a$ and $Y \in \mathfrak{k}$. With the identification (3.4) we then we compute

$$\begin{aligned} d\Phi_Z(b)(\tilde{Y}_b) &= \frac{d}{dt}\Big|_{t=0} \Phi_Z(\tilde{b}(\exp(tY).b) \\ &= \frac{d}{dt}\Big|_{t=0} \langle i\Phi(\tilde{b}(\exp(tY)b), Z\rangle \\ &= \frac{d}{dt}\Big|_{t=0} \langle i\Phi(\tilde{b}(b\exp(t\mathrm{Ad}(b^{-1})Y)), Z\rangle \\ &= \frac{d}{dt}\Big|_{t=0} \langle i\Im \log \tilde{a}(b\exp(t\mathrm{Ad}(b^{-1})Y)), Z\rangle \\ &= \langle i\Im \mathbf{pr}_{\mathfrak{a}_{\mathbb{C}}}(\mathrm{Ad}(b^{-1})Y), Z\rangle \\ &= \langle \mathbf{pr}_{\mathfrak{a}_{\mathbb{C}}}(\mathrm{Ad}(b^{-1})Y), Z\rangle. \end{aligned}$$

For the last equality we have used the fact that $\mathfrak{a} \perp \mathfrak{k}$ with respect to $\langle , \rangle$ (cf. Lemma 3.2).

On the other hand,

$$\begin{aligned} (\iota(\tilde{Z})\omega)_b(\tilde{Y}_b) &= \omega_b(\tilde{Z}_b, \tilde{Y}_b) = \langle \mathbf{pr}_{\mathfrak{k}_{\mathbb{C}}}(\mathrm{Ad}(b^{-1})Z), \mathrm{Ad}(b^{-1})Y\rangle \\ &= \langle \mathrm{Ad}(b^{-1})Z, \mathbf{pr}_{\mathfrak{b}_{\mathbb{C}}}(\mathrm{Ad}(b^{-1})Y)\rangle \\ &= \langle Z, \mathbf{pr}_{\mathfrak{b}_{\mathbb{C}}}(\mathrm{Ad}(b^{-1})Y)\rangle \\ &= \langle Z, \mathbf{pr}_{\mathfrak{a}_{\mathbb{C}}}(\mathrm{Ad}(b^{-1})Y)\rangle. \end{aligned}$$

The last two equations hold because $\mathrm{Ad}(b^{-1})Z \in Z + \mathfrak{n}_{\mathbb{C}}$, and $\langle \mathfrak{t}, \mathfrak{n}_{\mathbb{C}} \rangle = \{0\}$. □

3.4. **Symplectic proof of the complex convexity theorem.** As M_a is compact and the action of T on (M_a, ω) is Hamiltonian, the Atiyah-Guillemin-Sternberg convexity theorem [2, 5] asserts

$$\Phi(M_a) = \mathrm{conv}(\Phi(\mathrm{Fix}(M_a))) \ .$$

In this formula $\mathrm{conv}(\cdot)$ denotes the convex hull of $(\cdot)$ and $\mathrm{Fix}(M_a)$ stands for the T-fixed points in M_a. Standard structure theory implies that $\mathrm{Fix}(M_a) = \mathcal{W}.a$. We have thus proved:

Theorem 3.5. *Let G be a complex semisimple Lie group and $X \in \Omega$. Then*

$$\Im(\log \tilde{a}(K\exp(iX))) = \mathrm{conv}(\mathcal{W}.X) \ .$$

4. The complex convexity theorem for non-complex groups

For real groups the totally real K-orbits are no longer symplectic manifolds. However, they can be viewed as fixed point sets of an involution τ on the compact symplectic manifold M_a as introduced in Chapter 3. We will define τ and show that it is compatible with the action of T in a way that the symplectic convexity theorem from [6] can be applied.

Let $\mathfrak{g}_0$ be a non-compact real form of the complex Lie algebra $\mathfrak{g}$. It is no loss of generality if we assume that $\mathfrak{g}_0$ is θ-invariant. With $\mathfrak{k}_0 = \mathfrak{g}_0 \cap \mathfrak{k}$ and $\mathfrak{p}_0 = \mathfrak{g}_0 \cap \mathfrak{p}$ we then obtain a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ of $\mathfrak{g}_0$. We fix a maximal abelian subalgebra $\mathfrak{a}_0$ of $\mathfrak{p}_0$ which is contained in $\mathfrak{a}$. Write $\Sigma_0 = \Sigma(\mathfrak{g}_0, \mathfrak{a}_0)$ for the corresponding restricted root system and set

$$\Omega_0 = \{X \in \mathfrak{a}_0 : |\alpha(X)| < \frac{\pi}{2} \ \forall \alpha \in \Sigma_0\} \ .$$

As $\Sigma_0 = \Sigma|_{\mathfrak{a}_0} \backslash \{0\}$ we record that

$$\Omega_0 \subseteq \Omega \quad \text{and} \quad \Omega_0 = \Omega \cap \mathfrak{a}_0 \ . \tag{4.1}$$

It is no loss of generality to assume that $\Sigma_0^+ = \Sigma^+|_{\mathfrak{a}_0} \backslash \{0\}$ defines a positive system of Σ_0^+. We form the nilpotent Lie algebra $\mathfrak{n}_0 = \bigoplus_{\alpha \in \Sigma_0^+} \mathfrak{g}_0^\alpha$ and record that $\mathfrak{n}_0 = \mathfrak{g}_0 \cap \mathfrak{n}$.

The analytic subgroups of G with Lie algebras $\mathfrak{g}_0, \mathfrak{k}_0, \mathfrak{a}_0$ and $\mathfrak{n}_0$ will be denoted by G_0, K_0, A_0 and N_0. If $\Xi_0 = G_0 \exp(i\Omega_0)(K_0)_\mathbb{C}/(K_0)_\mathbb{C}$ denotes the crown of G_0/K_0, then (4.1) yields a holomorphic G_0-equivariant embedding

$$\Xi_0 \to \Xi \ . \tag{4.2}$$

As described at the end of Subsection 2.1, there exists a map $\log \tilde{a}_0 : \tilde{\Xi}_0 = (N_0)_\mathbb{C}(A_0)_\mathbb{C}(K_0)_\mathbb{C} \to (\mathfrak{a}_0)_\mathbb{C}$ with $\log \tilde{a}_0(\mathbf{1}) = 0$. Note that $\log \tilde{a}_0 = \log \tilde{a}|_{\tilde{\Xi}_0}$.

Let σ denote the Cartan involution on $\mathfrak{g}_0$. We also write σ for the doubly complex linear extension of σ to $\mathfrak{g}_\mathbb{C}$. Likewise θ also stands for the complex linear extension of θ to $\mathfrak{g}_\mathbb{C}$. We will be interested in the involution $\tau = \theta \circ \sigma = \sigma \circ \theta$ on $\mathfrak{g}_\mathbb{C}$ (which is the complex linear extension of the complex conjugation on $\mathfrak{g}$ with respect to $\mathfrak{g}_0$).
All these involutions on $\mathfrak{g}_\mathbb{C}$ can be lifted to involutions on $G_\mathbb{C}/K_\mathbb{C}$ and on $B_\mathbb{C}$, and we use the same letters to denote the lifts.

Recall the definition of the symplectic manifold $M_a = \tilde{b}(Ka)$ from Chapter 3. Notice that M_a is τ-invariant. The connected component of the τ-fixed point set which contains a is given by $Q_a = \tilde{b}(K_0 a)$. We also have a Hamiltonian action by the torus $T_0 = \exp(j\mathfrak{a}_0)$ on M_a.
The following lemma describes certain compatibility properties of the actions of T_0 and τ on M_a.

Lemma 4.1. *Consider the Hamiltonian torus action of $T_0 = \exp(j\mathfrak{a}_0)$ on M_a with momentum map $\Phi : M_a \to \mathfrak{t}_0^*$. Then the following assertions hold:*

(1) $t \circ \tau = \tau \circ t^{-1}$ *for all $t \in T_0$.*

(2) $\Phi \circ \tau = \Phi$.

(3) *Q_a is a Lagrangian submanifold of M_a.*

Proof. (1) : For $b=\tilde{b}(ka)\in M_a$ and $t\in T_0$,

$$t.\tau(b)=t.\tilde{b}(\tau(k)a)=\tilde{b}(t\tau(k)a)=\tilde{b}(\tau(t^{-1}k)a)=\tau(t^{-1}.b).$$

(2) : We need to introduce some additional notation.
Let

$$\mathfrak{n}^+:=\bigoplus_{\alpha\in\Sigma_0^+}\mathfrak{g}^\alpha\subseteq\mathfrak{n},\quad \mathfrak{n}^-:=\bigoplus_{\alpha\in\Sigma_0^+}\mathfrak{g}^{-\alpha},\quad\text{and}\quad \mathfrak{n}^0:=\bigoplus_{\alpha\in\Sigma^+\setminus\Sigma_0}\mathfrak{g}^\alpha.$$

We denote by $N_\mathbb{C}^+$, $N_\mathbb{C}^-$ and $N_\mathbb{C}^0$ the analytic subgroups of $G_\mathbb{C}$ with Lie algebras $\mathfrak{n}_\mathbb{C}^+$, $\mathfrak{n}_\mathbb{C}^-$ and $\mathfrak{n}_\mathbb{C}^0$, respectively. Notice that $\mathfrak{n}=\mathfrak{n}^0+\mathfrak{n}^+$ and therefore $N_\mathbb{C}=N_\mathbb{C}^+N_\mathbb{C}^0$. It is important to observe that $\tau(N_\mathbb{C}^+)=N_\mathbb{C}^+$ but $\tau(N_\mathbb{C}^0)\cap N_\mathbb{C}^0=\{\mathbf{1}\}$.

Write $\tilde{\mathfrak{t}}$ for a τ-invariant complement of $\mathfrak{a}_0+i\mathfrak{a}_0$ in $\mathfrak{a}_\mathbb{C}$. Then $\mathfrak{a}_\mathbb{C}=\mathfrak{a}_0+i\mathfrak{a}_0+\tilde{\mathfrak{t}}$. Let now $x=\tilde{k}a$ for some $\tilde{k}\in K$. Then $x,\tau(x)\in N_\mathbb{C}A_\mathbb{C}K_\mathbb{C}$ and

$$x=n_+n_0btk,\quad \tau(x)=n_+'n_0'b't'k' \tag{4.3}$$

with elements $n_+,n_+'\in N_\mathbb{C}^+,\ n_0,n_0'\in N_\mathbb{C}^0,\ b,b'\in\exp(\mathfrak{a}_0+i\mathfrak{a}_0),\ t,t'\in\exp(\tilde{\mathfrak{t}})$ and $k,k'\in K_\mathbb{C}$.

Clearly, (2) will be proved if we can show that $b^2=(b')^2$ (which forces $b=b'$ by the comments at the end of Subsection 2.1). This will be established in the sequel.

It follows from (4.3) that

$$\tau(x)=n_+'n_0'b't'k'=\tau(n_+)\tau(n_0)bt^{-1}\tau(k).$$

Since τ leaves $K_\mathbb{C}$ invariant and θ fixes each element of $K_\mathbb{C}$, we obtain

$$\tau(x)\theta(\tau(x))^{-1} = n_+'n_0'(b')^2\theta(n_0')^{-1}\theta(n_+')^{-1} \tag{4.4}$$

$$= \tau(n_+)\tau(n_0)b^2\theta(\tau(n_0))^{-1}\theta(\tau(n_+))^{-1}. \tag{4.5}$$

Notice that $n_0'(b')^2\theta(n_0')^{-1}$ and $\tau(n_0)b^2\theta(\tau(n_0))^{-1}$ belong to the reductive group $Z_{G_\mathbb{C}}(A_0)$, and recall that $\tau(N_\mathbb{C}^+)=N_\mathbb{C}^+$ and $\theta(N_\mathbb{C}^+)=N_\mathbb{C}^-$. Hence (4.4-5) combined with the Bruhat decomposition of $G_\mathbb{C}$ with respect to the parabolic subgroup $Z_{G_\mathbb{C}}(A_0)N_\mathbb{C}^+$ forces that $n_+'=\tau(n_+)$. But then we have

$$n_0'(b')^2\theta(n_0')^{-1}=\tau(n_0)b^2\theta(\tau(n_0))^{-1}$$

in $Z_{G_\mathbb{C}}(A_0)$. The components of $A_{0,\mathbb{C}}$, the center of $Z_{G_\mathbb{C}}(A_0)$, on both sides must coincide, therefore

$$(b')^2=b^2.$$

(3) : Consider $U,V\in\mathfrak{k}_0, k_0\in K_0$ and $b=\tilde{b}(k_0a)\in Q_a$. From the formula (3.1) for the symplectic form ω on M_a we get

$$\omega_b(\tilde{U}_b,\tilde{V}_b)=\langle\mathbf{pr}_{\mathfrak{k}_\mathbb{C}}(\mathrm{Ad}(b^{-1})U),\mathrm{Ad}(b^{-1})V\rangle.$$

Now, both $\mathbf{pr}_{\mathfrak{k}_\mathbb{C}}(\mathrm{Ad}(b^{-1})U)$ and $\mathrm{Ad}(b^{-1})V$ lie in $\mathfrak{g}_0+i\mathfrak{g}_0$. But for general elements $X_1,X_2,Y_1,Y_2\in\mathfrak{g}_0$ we have

$$\begin{aligned}\langle X_1+iX_2,Y_1+iY_2\rangle &= \Re\kappa(X_1+jX_2,Y_1+jY_2)-\Re\kappa(\bar{X}_1+j\bar{X}_2,\bar{Y}_1+j\bar{Y}_2)\\ &= \Re\kappa(X_1,Y_1)-\Re\kappa(X_2,Y_2)-\Re\kappa(\bar{X}_1,\bar{Y}_1)+\Re\kappa(\bar{X}_2,\bar{Y}_2)\\ &\quad+\Re\kappa(X_1,jY_2)+\Re\kappa(jX_2,Y_1)-\Re\kappa(\bar{X}_1,j\bar{Y}_2)-\Re\kappa(j\bar{X}_2,\bar{Y}_1)\\ &= 0\end{aligned}$$

The last equality is due to the invariance of κ and the fact that $X_1,X_2,Y_1,Y_2\in\mathfrak{g}_0$. This shows that $\omega_b(\tilde{U}_b,\tilde{V}_b)=0$, i.e. $T_b(Q_a)$ is isotropic. □

We recall the following symplectic convexity theorem [6].

Theorem 4.2. *Let M be a compact connected symplectic manifold with Hamiltonian torus action $T \times M \to M$ and momentum map $\Phi : M \to \mathfrak{t}^*$. In addition, let $\tau : M \to M$ be an involutive diffeomorphism with fixed point set Q such that*

(1) $t \circ \tau = \tau \circ t^{-1}$ *for all* $t \in T$.

(2) $\Phi \circ \tau = \Phi$.

(3) Q *is a Lagrangian submanifold of* M.

Denote the T-fixed subsets of M and Q by $\mathrm{Fix}(M)$ *and* $\mathrm{Fix}(Q)$*, respectively. Then,*

$$\Phi(Q) = \Phi(M) = \mathrm{conv}(\Phi(\mathrm{Fix(M)})) = \mathrm{conv}(\Phi(\mathrm{Fix(Q)})).$$

Moreover, the same assertions hold if Q is replaced with any of its connected components.

With this result at hand we are now able to prove the complex convexity result for non complex groups. Write $\mathcal{W}_0$ for the Weyl group of Σ_0.

Theorem 4.3. *Let G_0 be a non-compact connected semisimple Lie group with Lie algebra $\mathfrak{g}_0$. Fix an element $X \in \Omega_0$. Then*

$$\Im \log \tilde{a}_0(K_0 \exp(iX)) = \mathrm{conv}(\mathcal{W}_0.\mathrm{X}) \ . \tag{4.6}$$

Proof. Define $a = \exp(iX)$. The left hand side in equality (4.6) coincides with $\Phi(Q_a)$ where $\Phi = \Im \circ \log \circ \tilde{a}$ is the momentum map on M_a. Lemma 4.1 says that conditions (1)-(3) in Theorem 4.2 are satisfied. Therefore,

$$\Im \log \tilde{a}_0(K_0 \exp(iX)) = \Phi(Q_a) = \mathrm{conv}(\Phi(\mathrm{Fix(Q_a)})).$$

Standard structure theory shows that $\mathrm{Fix}(Q_a) = \tilde{b}(\mathcal{W}_0 \exp(iX))$. This implies $\Phi(\mathrm{Fix}(Q_a)) = \mathcal{W}_0.X$, and finishes the proof. □

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